Distributed Composite Quantization

Supplementary Material

1 Outline

This document proves the convergence of ADMM for the problem Equation (11) in the AAAI submission Distributed Composite Quantization. The proof is based on the SIAM paper [4], in which it is proved that ADMM can reach convergence for a set of nonconvex problems including the consensus and sharing problems that satisfies certain assumptions. As stated in the abstract of [4], "We show that the classical ADMM converges to the set of stationary solutions, provided that the penalty parameter in the augmented Lagrangian is chosen to be sufficiently large". In Section 2.2 of [4], Hong *et al.* proposes the Flexible ADMM for the nonconvex global consensus problem whose each sub-problem can be exactly solved. In Section 2.3, they generalize the Flexible ADMM as Flexible Proximal ADMM for the case that sub-problem cannot be exactly solved. In Section 3, Hong *et al.* considers the nonconvex sharing problem.

As stated in Page 4 in [4], the Flexible ADMM is "a more general version of ADMM which includes the classical ADMM as a special case". From page 5 to page 15 in [4], Hong *et al.* have proven the convergence of the Flexible ADMM for the nonconvex global consensus problem whose each sub-problem can be exactly solved.

The optimization problem in our AAAI submission falls into the framework of Section 2.2 of [4]. Therefore, we could prove the convergence of our optimization problem by showing that:

1. The optimization problem in our AAAI submission is a nonconvex global consensus problem, whose each sub-problem in the augmented Lagrangian can be exactly solved.
2. The certain assumptions required by [4] are satisfied.

The proof of convergence is organized as following:

- In Section 2, we introduce the nonconvex global consensus problem considered in [4], and the required assumptions for the convergence of ADMM.
- In Section 3, we reformulate the optimization problem in our AAAI manuscript into the form of the nonconvex global consensus problem.
- In Section 4, we show that our optimization problem satisfies the certain assumptions required by [4].

2 Nonconvex Global consensus problem

The nonconvex global consensus problem considered in [4] is defined as

$$\begin{aligned} \min \quad & \sum_{k=1}^{K} g_k(x_k) + h(x_0) \\ \text{s.t.} \quad & x_k = x_0, \forall k = 1, \dots, K, x_0 \in X, \end{aligned} \tag{1}$$

where g_k's are a set of smooth, possibly nonconvex functions, $h(x)$ is a convex nonsmooth regularization term, X is a convex set. Note that $h(x)$ does not need to be strictly convex. This problem is related to the convex global consensus problem discussed heavily in [2], but with the important difference that g_k's can be nonconvex.

The augmented Lagrangian for Equation 1 is given by:

$$L(\{x_k\}, x_0; y) = \sum_{k=1}^{K} g_k(x_k) + h(x_0) + \sum_{k=1}^{K} \langle y_k, x_k - x_0 \rangle + \sum_{k=1}^{K} \frac{\rho_k}{2} ||x_k - x_0||^2 \tag{2}$$

In Section 2.2 of [4], Hong *et al.* proposes to solve Equation 2 with the Flexible ADMM, a more general version of ADMM which includes the classical ADMM as a special case. The sub-problem of Equation 2 could be easily written as:

$$\phi(x_k) = g_k(x_k) + \langle y_k, x_k - x_0 \rangle + \frac{\rho_k}{2} ||x_k - x_0||^2 \tag{3}$$

As long as certain assumptions are satisfied, the convergence of the Flexible ADMM can be theoretically guaranteed.

These assumptions are:

1. There exists a positive constant $L_k > 0$ such that

   $$||\nabla_k g_k(x_k) - \nabla_k g_k(z_k)|| \le L_k ||x_k - z_k||, \forall x_k, z_k, k = 1, 2, \dots, K. \tag{4}$$

   Moreover, h is convex (possible nonsmooth); X is a closed convex set.

2. For all k, the penalty parameter ρ_k is chosen large enough such that:

   (a) For all k, the sub-problem $\phi(x_k)$ is strongly convex with modulus $\gamma_k(\rho_k)$

   (b) For all k, $\rho\gamma_k(\rho) > 2{L_k}^2$ and $\rho \ge L_k$.

3. The original problem Eq.(1) is lower bounded, *i.e.*, $\min \sum_{k=1}^{K} g_k(x_k) + h(x_0) > -\infty$

Remarks:

1. We use L-BFGS to solve each sub-problem $\phi(x_k)$ in our AAAI submission, and Therefore, $\phi(x_k)$ will be exactly solved as long as it is convex. The assumption 2(a) has assumes $\phi(x_k)$ to be strictly convex.

2. We use the same ρ for all K sub-problems in our submitted manuscript, one may consider this as $\rho = \rho_1 = \rho_2 = \cdots = \rho_K$.

3 Problem Reformulation

Recall the optimization problem in our AAAI submission. We apply the classical ADMM to optimize the Equation (10) in our submitted manuscript, which is as follows:

$$\begin{aligned} \min_{\{C^s\}} \quad & \sum_{s=1}^{P} ||X^s - C^s B^s||_F^2 + \mu \sum_{s=1}^{P} \sum_{n=1}^{N^s} (\sum_{i \neq j}^{M} {b_{ni}^s}^T C_i^{sT} C_j^s b_{nj}^s - \epsilon^s)^2 \\ \text{s.t.} \quad & C^s = C^{s'}, s' \in \mathcal{N}(s), \forall s \in \{1, 2, \dots, P\}. \end{aligned} \tag{5}$$

For consistency with Section 2, we replace s by k, P by K, and rewrite Equation 5 as below

$$\begin{aligned} \min_{\{C^k\}} \quad & \sum_{k=1}^{K} ||X^k - C^k B^k||_F^2 + \mu \sum_{k=1}^{K} \sum_{n=1}^{N^k} (\sum_{i \neq j}^{M} {b_{ni}^k}^T C_i^{kT} C_j^k b_{nj}^k - \epsilon^k)^2 \\ \text{s.t.} \quad & C^k = C^{k'}, k' \in \mathcal{N}(k), \forall k \in \{1, 2, \dots, K\}. \end{aligned} \tag{6}$$

By defining the functions $g_k(C^k)$ and $h(C)$ as:

$$\begin{aligned} g_k(C^k) &= ||X^k - C^k B^k||_F^2 + \mu \sum_{n=1}^{N^k} (\sum_{i \neq j}^{M} {b_{ni}^k}^T C_i^{kT} C_j^k b_{nj}^k - \epsilon^k)^2, \\ h(C) &= 0, \end{aligned} \tag{7}$$

we could write Equation 6 into the form of nonconvex global consensus problem:

$$\begin{aligned} \min \quad & \sum_{k=1}^{K} g_k(C^k) + h(C^0) \\ \text{s.t.} \quad & \mathrm{C}^k = \mathrm{C}^0, \forall k \in \{1, 2, \dots, K\}. \end{aligned} \tag{8}$$

As mentioned in Section 2, function $h(C)$ does not need to be strictly convex, thus we define $h(C) = 0$ to introduce the auxiliary variable C^0. Note that we do not have C^0 in the original formulation Eq. (5). Since the change of C^0 will not increase the objective, C^0 can be "implicitly" updated to any value to keep the consensus between the true variables $\{\mathrm{C}^s\}$, though C^0 does not exist in Eq. (5) in fact. The augmented Lagrangian of Eq. (8) is:

$$L(\{C^k\}, \{\Lambda^k\}) = \sum_{k=1}^{K} g_k(C^k) + h(C^0) + \sum_{k=1}^{K} \mathrm{tr}\big({\Lambda^k}^T (C^k - C^0)\big) + \frac{\rho}{2} \sum_{k=1}^{K} ||C^k - C^0||_F^2, \tag{9}$$

where Λ^k serves the same role as y_k in Equation 2. Consequently, the sub-problem for 9 is:

$$\phi(C^k) = g_k(C^k) + \mathrm{tr}\big({\Lambda^k}^T (C^k - C^0)\big) + \frac{\rho}{2} ||C^k - C^0||_F^2, \tag{10}$$

Next, we show that our problem satisfies the assumptions listed in Section 2.

4 Proof

4.1 Proof of Assumption 1

The assumption 1 is:

- There exists a positive constant $L_k > 0$ such that

$$||\nabla_k g_k(x_k) - \nabla_k g_k(z_k)|| \leq L_k ||x_k - z_k||, \forall x_k, z_k, k = 1, 2, \ldots, K. \tag{11}$$

  Moreover, h is convex (possible nonsmooth); X is a closed convex set.

Now we turn to our specific problem, we have

$$g_k(C^k) = ||X^k - C^k B^k||_F^2 + \mu \sum_{n=1}^{N^k} (\sum_{i \neq j}^{M} {b_{ni}^k}^T {C_i^k}^T C_j^k b_{nj}^k - \epsilon^k)^2, \tag{12}$$

Since all $g_k(C^k)$'s in our AAAI optimization problem have the same form, we remove the k superscript and subscript for clarification. So Equation 12 can be simplified as:

$$g(C) = ||X - CB||_F^2 + \mu \sum_{n=1}^{N} (\sum_{i \neq j}^{M} {b_{ni}}^T {C_i}^T C_j b_{nj} - \epsilon)^2, \tag{13}$$

To compute $\nabla g(C)$, we first compute the partial derivative to each sub-matrix of C. For $m = 1, 2, \ldots, M$, we have

$$\frac{\partial g(C)}{\partial C_m} = 2 \sum_{n=1}^{N} \big[(\sum_{l=1}^{M} C_l b_{nl} - X_n) {b_{nm}}^T + 2\mu (\sum_{i \neq j}^{M} {b_{ni}}^T {C_i}^T C_j b_{nj} - \epsilon)(\sum_{l \neq m}^{M} C_l b_{nl}) {b_{nm}}^T \big]. \tag{14}$$

The $\nabla g(C)$ is the stack of Equation 14 for $m = 1, 2, \ldots, M$. To prove Inequality 11, we have to show that:

- There exists a positive constant $L > 0$ for any matrix Y and Z such that

$$||\nabla g(Y) - \nabla g(Z)||_F \leq L ||Y - Z||_F \tag{15}$$

  Moreover, h is convex (possible nonsmooth); Y and Z are sampled from a closed convex set.

Remarks:

1. Since $f(x) = x^2$ is monotonically increasing when $x \geq 0$, Equation 15 is sufficient and necessary to

$$||\nabla g(Y) - \nabla g(Z)||_F^2 \leq L^2 ||Y - Z||_F^2 \tag{16}$$

2. We define $h(C) = 0$, which is clearly convex (although not strictly).

3. For our specific problem, we use the composition of elments of C to approximate feature vectors. For the three datasets MNIST [7], LabelMe22K [12] and SIFT1M [5], the feature vectors are raw pixel value, GIST feature, and SIFT feature, which are all non-nagetive. During the training phase of the algorithm in our AAAI submission, it is reasonable to assume that any entry of any feature vector is less or equal to some positive number $\mathcal{U}$. For example, given the training set X_{train}, one may easily find $\mathcal{U}$ by executing a Matlab command $max(max(X_{train}))$. Thus, any entry of Y and Z should be less or equal to $\mathcal{U}$. The matrix Y and Z are sampled from a convex set $\mathcal{C} = [-\mathcal{U}, \mathcal{U}]^{d \times MK}$.

By moving the term $||Y - Z||_F^2$ in Equation 16 to the left side, we have

$$\frac{||\nabla g(Y) - \nabla g(Z)||_F^2}{||Y - Z||_F^2} \leq L^2 \tag{17}$$

Therefore, the ultimate target is to prove that the left-hand term is upper-bounded, and set L^2 to be the upper bound. First, let us analyse the scalar term $2\mu(\sum_{i \neq j}^{M} {b_{ni}}^T {C_i}^T C_j b_{nj} - \epsilon)$ in Equation 14. In our AAAI submission, b_{ni} is defined as a one-hot vector, *i.e.*, only one entry of b_{ni} is 1, other entries are 0, thus,

$$\begin{aligned} -d\mathcal{U}^2 \leq {b_{ni}}^T {C_i}^T C_j b_{nj} = (b_{ni} C_i)^T C_j b_{nj} \leq d\mathcal{U}^2 \\ -dM^2\mathcal{U}^2 \leq \sum_{i \neq j}^{M} {b_{ni}}^T {C_i}^T C_j b_{nj} \mathcal{U}^2 \leq dM^2\mathcal{U}^2 \end{aligned} \tag{18}$$

ϵ is a variable to minimize $(\sum_{i \neq j}^{M} {b_{ni}}^T {C_i}^T C_j b_{nj} - \epsilon)$, thus,

$$-dM^2\mathcal{U}^2 \leq \epsilon \leq dM^2\mathcal{U}^2. \tag{19}$$

Therefore, the scalar term $2\mu(\sum_{i \neq j}^{M} {b_{ni}}^T {C_i}^T C_j b_{nj} - \epsilon)$ is bounded by:

$$-4\mu d M^2\mathcal{U}^2 \leq 2\mu(\sum_{i \neq j}^{M} {b_{ni}}^T {C_i}^T C_j b_{nj} - \epsilon) \leq 4\mu d M^2\mathcal{U}^2 \tag{20}$$

Also, we will use the following three properties of Frobenius norm for the proof next:

$$\begin{aligned} ||A + B||_F^2 &= ||A||_F^2 + ||B||_F^2 + 2\text{tr } (A^T B) \\ &\leq ||A||_F^2 + ||B||_F^2 + 2||A||_F||B||_F \\ &\leq (||A||_F + ||B||_F)^2 \end{aligned} \tag{21}$$

and

$$||AB||_F^2 \leq ||A||_F^2 ||B||_F^2, \tag{22}$$

and for any one-hot vector b and vector I whose all entries are 1,

$$||Ab||_F^2 \leq ||AI||_F^2 \tag{23}$$

in this case, $|| \cdot ||_F^2$ is equivalent to $|| \cdot ||_2^2$.

Consider the numerator of the left hand of Equation 17,

$$
\begin{aligned}
& ||\nabla g(Y) - \nabla g(Z)||_F^2 \\
&= \sum_{m=1}^{M} ||\frac{\partial g(Y)}{\partial Y_m} - \frac{\partial g(Z)}{\partial Z_m}||_F^2 \\
&= \sum_{m=1}^{M} \Big|\Big| 2\sum_{n=1}^{N} \Big[(\sum_{l=1}^{M} Y_l b_{nl} - X_n) {b_{nm}}^T + 2\mu(\sum_{i\neq j}^{M} {b_{ni}}^T {Y_i}^T Y_j b_{nj} - \epsilon)(\sum_{l\neq m}^{M} Y_l b_{nl}) {b_{nm}}^T \\
&\qquad - (\sum_{l=1}^{M} Z_l b_{nl} - X_n) {b_{nm}}^T - 2\mu(\sum_{i\neq j}^{M} {b_{ni}}^T {Z_i}^T Z_j b_{nj} - \epsilon)(\sum_{l\neq m}^{M} Z_l b_{nl}) {b_{nm}}^T \Big] \Big|\Big|_F^2 \\
&= 4\sum_{m=1}^{M} \Big|\Big| \sum_{n=1}^{N} \Big[(\sum_{l=1}^{M} Y_l b_{nl} - X_n) + 2\mu(\sum_{i\neq j}^{M} {b_{ni}}^T {Y_i}^T Y_j b_{nj} - \epsilon)(\sum_{l\neq m}^{M} Y_l b_{nl}) \\
&\qquad - (\sum_{l=1}^{M} Z_l b_{nl} - X_n) - 2\mu(\sum_{i\neq j}^{M} {b_{ni}}^T {Z_i}^T Z_j b_{nj} - \epsilon)(\sum_{l\neq m}^{M} Z_l b_{nl}) \Big] {b_{nm}}^T \Big|\Big|_F^2 \\
&= 4\sum_{m=1}^{M} \Big|\Big| \sum_{n=1}^{N} \Big[\sum_{l=1}^{M} (Y_l b_{nl} - Z_l b_{nl}) + 2\mu(\sum_{i\neq j}^{M} {b_{ni}}^T {Y_i}^T Y_j b_{nj} - \epsilon)(\sum_{l\neq m}^{M} Y_l b_{nl}) \\
&\qquad - 2\mu(\sum_{i\neq j}^{M} {b_{ni}}^T {Z_i}^T Z_j b_{nj} - \epsilon)(\sum_{l\neq m}^{M} Z_l b_{nl}) \Big] {b_{nm}}^T \Big|\Big|_F^2 \\
&\leq 4\sum_{m=1}^{M} \Big|\Big| \sum_{n=1}^{N} \Big[\sum_{l=1}^{M} (Y_l b_{nl} - Z_l b_{nl}) + 4\mu d M^2 \mathcal{U}^2 (\sum_{l\neq m}^{M} Y_l b_{nl}) - 4\mu d M^2 \mathcal{U}^2 (\sum_{l\neq m}^{M} Z_l b_{nl}) \Big] {b_{nm}}^T \Big|\Big|_F^2 \\
&\leq 4\sum_{m=1}^{M} \Big|\Big| \sum_{n=1}^{N} \Big[\sum_{l=1}^{M} (Y_l b_{nl} - Z_l b_{nl}) + 4\mu d M^2 \mathcal{U}^2 (\sum_{l=1}^{M} Y_l b_{nl}) - 4\mu d M^2 \mathcal{U}^2 (\sum_{l=1}^{M} Z_l b_{nl}) \Big] {b_{nm}}^T \Big|\Big|_F^2 \\
&\leq 4\sum_{m=1}^{M} \Big|\Big| \sum_{n=1}^{N} \Big[(1 + 4\mu d M^2 \mathcal{U}^2) \sum_{l=1}^{M} (Y_l b_{nl} - Z_l b_{nl}) \Big] {b_{nm}}^T \Big|\Big|_F^2 \\
&\leq 4(1 + 4\mu d M^2 \mathcal{U}^2)^2 \sum_{m=1}^{M} \Big|\Big| \sum_{n=1}^{N} \Big[\sum_{l=1}^{M} (Y_l - Z_l) b_{nl} {b_{nm}}^T \Big] \Big|\Big|_F^2 \\
&\leq 4(1 + 4\mu d M^2 \mathcal{U}^2)^2 \sum_{m=1}^{M} N^2 \max_n \Big|\Big| \Big[\sum_{l=1}^{M} (Y_l - Z_l) b_{nl} {b_{nm}}^T \Big] \Big|\Big|_F^2 \\
&\leq 4N^2(1 + 4\mu d M^2 \mathcal{U}^2)^2 M \max_n \Big|\Big| \sum_{l=1}^{M} (Y_l - Z_l) b_{nl} \Big|\Big|_F^2 \\
&\leq 4N^2(1 + 4\mu d M^2 \mathcal{U}^2)^2 M \Big|\Big| \sum_{l=1}^{M} (Y_l - Z_l) \Big|\Big|_F^2 \\
&\leq 4N^2(1 + 4\mu d M^2 \mathcal{U}^2)^2 M (\sum_{l=1}^{M} ||Y_l - Z_l||_F^2)
\end{aligned}
\tag{24}
$$

Now let us turn to the denominator left hand of Equation 17, by definition we have

$$||Y - Z||_F^2 = \sum_{l=1}^{M} ||Y_l - Z_l||_F^2 \tag{25}$$

By combining Equation 17, 24 and 25, we reach the following conclusion:

$$\begin{aligned} \frac{||\nabla g(Y) - \nabla g(Z)||_F^2}{||Y - Z||_F^2} &\leq \frac{4N^2(1 + 4\mu d M^2 \mathcal{U}^2)^2 M (\sum_{l=1}^{M} ||Y_l - Z_l||_F^2)}{\sum_{l=1}^{M} ||Y_l - Z_l||_F^2} \\ &\leq 4N^2(1 + 4\mu d M^2 \mathcal{U}^2)^2 M \end{aligned} \tag{26}$$

thus, the desired L will be

$$L = \sqrt{4N^2(1 + 4\mu d M^2 \mathcal{U}^2)^2 M} = 2N(1 + 4\mu d M^2 \mathcal{U}^2)\sqrt{M} \tag{27}$$

4.2 Proof of Assumption 2(a)

The assumption 2(a) is:

- For all k, the penalty parameter ρ_k is chosen large enough such that:
  - For all k, the sub-problem $\phi(x_k)$ is strongly convex with modulus $\gamma_k(\rho_k)$

Remarks

- A differentiable function $f(\cdot)$ is called strongly convex with parameter $m > 0$, if for any input x, y and $t \in [0, 1]$, the following inequality holds :

$$f\big(tx + (1-t)y\big) \leq tf(x) + (1-t)f(y) - \frac{1}{2}mt(1-t)||x - y||_2^2 \tag{28}$$

- Modulus measures how convex a function is.

To our specific problem, our $\phi(x_k)$ can be written as:

$$\begin{aligned} g_k(C^k) &= ||X^k - C^k B^k||_F^2 + \mu \sum_{n=1}^{N^k} (\sum_{i \neq j}^{M} {b_{ni}^k}^T {C_i^k}^T C_j^k b_{nj}^k - \epsilon^k)^2, \\ \phi(C^k) &= g_k(C^k) + \mathrm{tr}\big({\Lambda^k}^T (C^k - C^0)\big) + \frac{\rho}{2}||C^k - C^0||_F^2, \end{aligned} \tag{29}$$

We use the same ρ for all K sub-problems in our submitted manuscript, one may consider this as $\rho = \rho_1 = \rho_2 = \cdots = \rho_K$. Since all our $g_k(C^k)$ and $\phi(C^k)$ have the same form, we remove the superscript k for clarification. Thus Equation 29 could be simplified as:

$$\begin{aligned} g(C) &= ||X - CB||_F^2 + \mu \sum_{n=1}^{N} (\sum_{i \neq j}^{M} {b_{ni}}^T {C_i}^T C_j b_{nj} - \epsilon)^2, \\ \phi(C) &= g(C) + \mathrm{tr}\big(\Lambda^T (C - C^0)\big) + \frac{\rho}{2}||C - C^0||_F^2, \end{aligned} \tag{30}$$

To satisfy the assumption 2(a), we need to prove that for any matrices $Y, Z \in [-\mathcal{U}, \mathcal{U}]^{d\times MK}$, there exist a $m > 0$ that makes the following inequality hold:

$$\phi(tY + (1-t)Z) \leq t\phi(Y) + (1-t)\phi(Z) - \frac{1}{2}mt(1-t)||Y - Z||_F^2 \tag{31}$$

which is equivalent to

$$2\frac{t\phi(Y) + (1-t)\phi(Z) - \phi(tY + (1-t)Z)}{t(1-t)||Y - Z||_F^2} \geq m \tag{32}$$

thus, our target is to find a positive lower bound for the left hand of Equation 32.

Now let us consider the numerator of the left hand of Equation 32,

$$\begin{aligned}
&t\phi(Y) + (1-t)\phi(Z) - \phi(tY + (1-t)Z)\\
&= t\big[g(Y) + \mathrm{tr}\big(\Lambda^T(Y - C^0)\big) + \frac{\rho}{2}||Y - C^0||_F^2\big] + (1-t)\big[g(Z) + \mathrm{tr}\big(\Lambda^T(Z - C^0)\big) + \frac{\rho}{2}||Z - C^0||_F^2\big]\\
&\quad - \big[g(tY + (1-t)Z) + \mathrm{tr}\big(\Lambda^T(tY + (1-t)Z - C^0)\big) + \frac{\rho}{2}||tY + (1-t)Z - C^0||_F^2\big]\\
&= tg(Y) + (1-t)g(Z) - g(tY + (1-t)Z)\\
&\quad + t\ \mathrm{tr}\big(\Lambda^T(Y - C^0)\big) + (1-t)\mathrm{tr}\big(\Lambda^T(Z - C^0)\big) - \mathrm{tr}\big(\Lambda^T(tY + (1-t)Z - C^0)\big)\\
&\quad + t\frac{\rho}{2}||Y - C^0||_F^2 + (1-t)\frac{\rho}{2}||Z - C^0||_F^2 - \frac{\rho}{2}||tY + (1-t)Z - C^0||_F^2\\
&= tg(Y) + (1-t)g(Z) - g(tY + (1-t)Z)\\
&\quad + t\ \mathrm{tr}\big(\Lambda^TY\big) + (1-t)\mathrm{tr}\big(\Lambda^TZ\big) - \mathrm{tr}\big(\Lambda^TtY\big) - \mathrm{tr}\big(\Lambda^T(1-t)Z\big) + 2\mathrm{tr}(C^{0^T}C^0)\\
&\quad + t\frac{\rho}{2}||Y||_F^2 + t\frac{\rho}{2}||C^0||_F^2 - t\rho||Y||_F||C^0||_F + (1-t)\frac{\rho}{2}||Z||_F^2 + (1-t)\frac{\rho}{2}||C^0||_F^2 - (1-t)\rho||Z||_F||C^0||_F\\
&\quad - t^2\ \frac{\rho}{2}||Y||_F^2 - (1-t)^2\ \frac{\rho}{2}||Z||_F^2 - \rho||tY||_F||(1-t)Z||_F + \rho||tY||_F||C^0||_F + \rho||(1-t)Z||_F||C^0||_F\\
&= tg(Y) + (1-t)g(Z) - g(tY + (1-t)Z) + 2\mathrm{tr}(C^{0^T}C^0)\\
&\quad + t\frac{\rho}{2}||Y||_F^2 - t\rho||Y||_F||C^0||_F - (1-t)\rho||Z||_F||C^0||_F\\
&\quad - t^2\ \frac{\rho}{2}||Y||_F^2 + (t - t^2)\ \frac{\rho}{2}||Z||_F^2 - \rho||tY||_F||(1-t)Z||_F + \rho||tY||_F||C^0||_F + \rho||(1-t)Z||_F||C^0||_F\\
&= tg(Y) + (1-t)g(Z) - g(tY + (1-t)Z)\\
&\quad + 2\mathrm{tr}(C^{0^T}C^0) + t\frac{\rho}{2}||Y||_F^2 + t\frac{\rho}{2}||Z||_F^2 - t\rho||Y||_F||C^0||_F - (1-t)\rho||Z||_F||C^0||_F\\
&\quad - t^2\ \frac{\rho}{2}||Y||_F^2 - t^2\ \frac{\rho}{2}||Z||_F^2 - \rho\sqrt{t}\sqrt{1-t}||Y||_F||Z||_F + \rho\sqrt{t}||Y||_F||C^0||_F + \rho\sqrt{1-t}||Z||_F||C^0||_F
\end{aligned} \tag{33}$$

Now, let us focus to the row that involves $g(\cdot)$,

$$
\begin{aligned}
& tg(Y)+(1-t)g(Z)-g(tY+(1-t)Z) \\
&= t\Big[||X-YB||_F^2+\mu\sum_{n=1}^{N}(\sum_{i\neq j}^{M} {b_{ni}}^T {Y_i}^T Y_j^k b_{nj}-\epsilon)^2\Big]+(1-t)\Big[||X-ZB||_F^2+\mu\sum_{n=1}^{N}(\sum_{i\neq j}^{M} {b_{ni}}^T {Z_i}^T Z_j b_{nj}-\epsilon)^2\Big] \\
&\quad -||X-\big(t\,Y+(1-t)\,Z\big)B||_F^2-\mu\sum_{n=1}^{N}(\sum_{i\neq j}^{M} {b_{ni}}^T \big(t\,Y+(1-t)\,Z\big)_i^T\big(t\,Y+(1-t)\,Z\big)_j b_{nj}-\epsilon)^2 \\
&= t\Big[||X||_F^2+||YB||_F^2-2\mathrm{tr}(X^TYB)+\mu\sum_{n=1}^{N}\big[\big(\sum_{i\neq j}^{M} {b_{ni}}^T {Y_i}^T Y_j b_{nj}\big)^2-2\epsilon\big(\sum_{i\neq j}^{M} {b_{ni}}^T {Y_i}^T Y_j b_{nj}\big)+\epsilon^2\big]\Big] \\
&\quad +(1-t)\Big[||X||_F^2+||ZB||_F^2-2\mathrm{tr}(X^TZB)+\mu\sum_{n=1}^{N}\big[\big(\sum_{i\neq j}^{M} {b_{ni}}^T {Z_i}^T Z_j b_{nj}\big)^2-2\epsilon\big(\sum_{i\neq j}^{M} {b_{ni}}^T {Z_i}^T Z_j b_{nj}\big)+\epsilon^2\big]\Big] \\
&\quad -||X||_F^2-||\big(t\,Y+(1-t)\,Z\big)B||_F^2+2\mathrm{tr}\Big(X^T\big(t\,Y+(1-t)\,Z\big)B\Big) \\
&\quad -\mu\sum_{n=1}^{N}\big[\big(\sum_{i\neq j}^{M} {b_{ni}}^T\big(t\,Y+(1-t)\,Z\big)_i^{\,T}\big(t\,Y+(1-t)\,Z\big)_j b_{nj}\big)^2 \\
&\quad -2\epsilon\big(\sum_{i\neq j}^{M} {b_{ni}}^T\big(t\,Y+(1-t)\,Z\big)_i^{\,T}\big(t\,Y+(1-t)\,Z\big)_j b_{nj}\big)+\epsilon^2\big] \\
&= t\Big[||Y||_F^2+2\mathrm{tr}(Y^TB)+||B||_F^2+\mu\sum_{n=1}^{N}\big[\big(\sum_{i\neq j}^{M} {b_{ni}}^T {Y_i}^T Y_j^k b_{nj}\big)^2-2\epsilon\big(\sum_{i\neq j}^{M} {b_{ni}}^T {Y_i}^T Y_j^k b_{nj}\big)\big]\Big] \\
&\quad +(1-t)\Big[||Z||_F^2+2\mathrm{tr}(Z^TB)+||B||_F^2+\mu\sum_{n=1}^{N}\big[\big(\sum_{i\neq j}^{M} {b_{ni}}^T {Z_i}^T Z_j^k b_{nj}\big)^2-2\epsilon\big(\sum_{i\neq j}^{M} {b_{ni}}^T {Z_i}^T Z_j^k b_{nj}\big)\big]\Big] \\
&\quad -||\big(t\,Y+(1-t)\,Z\big)||_F^2-2\mathrm{tr}\Big(\big(t\,Y+(1-t)\,Z\big)^TB\Big)-||B||_F^2 \\
&\quad -\mu\sum_{n=1}^{N}\big[\big(\sum_{i\neq j}^{M} {b_{ni}}^T\big(t\,Y+(1-t)\,Z\big)_i^{\,T}\big(t\,Y+(1-t)\,Z\big)_j b_{nj}\big)^2 \\
&\quad -2\epsilon\big(\sum_{i\neq j}^{M} {b_{ni}}^T\big(t\,Y+(1-t)\,Z\big)_i^{\,T}\big(t\,Y+(1-t)\,Z\big)_j b_{nj}\big)\big] \\
&= t\Big[||Y||_F^2+\mu\sum_{n=1}^{N}\big[\big(\sum_{i\neq j}^{M} {b_{ni}}^T {Y_i}^T Y_j^k b_{nj}\big)^2-2\epsilon\big(\sum_{i\neq j}^{M} {b_{ni}}^T {Y_i}^T Y_j^k b_{nj}\big)\big]\Big] \\
&\quad +(1-t)\Big[||Z||_F^2+\mu\sum_{n=1}^{N}\big[\big(\sum_{i\neq j}^{M} {b_{ni}}^T {Z_i}^T Z_j^k b_{nj}\big)^2-2\epsilon\big(\sum_{i\neq j}^{M} {b_{ni}}^T {Z_i}^T Z_j^k b_{nj}\big)\big]\Big]-||\big(t\,Y+(1-t)\,Z\big)||_F^2 \\
&\quad -\mu\sum_{n=1}^{N}\big[\big(\sum_{i\neq j}^{M} {b_{ni}}^T\big(t\,Y+(1-t)\,Z\big)_i^{\,T}\big(t\,Y+(1-t)\,Z\big)_j b_{nj}\big)^2 \\
&\quad -2\epsilon\big(\sum_{i\neq j}^{M} {b_{ni}}^T\big(t\,Y+(1-t)\,Z\big)_i^{\,T}\big(t\,Y+(1-t)\,Z\big)_j b_{nj}\big)\big]
\end{aligned}
\tag{34}
$$

By putting Equation 33 and 34 together, we have

$$
\begin{aligned}
& t\phi(Y) + (1-t)\phi(Z) - \phi(tY + (1-t)Z) \\
= & \, t||Y||_F^2 + t\mu \sum_{n=1}^{N} \big[\big(\sum_{i \neq j}^{M} {b_{ni}}^T Y_i^T Y_j^k b_{nj} \big)^2 - 2\epsilon \big(\sum_{i \neq j}^{M} {b_{ni}}^T Y_i^T Y_j^k b_{nj} \big) \big] \\
& + (1-t)||Z||_F^2 + (1-t)\mu \sum_{n=1}^{N} \big[\big(\sum_{i \neq j}^{M} {b_{ni}}^T Z_i^T Z_j^k b_{nj} \big)^2 - 2\epsilon \big(\sum_{i \neq j}^{M} {b_{ni}}^T Z_i^T Z_j^k b_{nj} \big) \big] \\
& - t^2||Y||_F^2 - (1-t)^2||Z||_F^2 - 2t\mathrm{tr}(Y^T Z) + 2t^2\mathrm{tr}(Y^T Z) \\
& - \mu \sum_{n=1}^{N} \big[\big(\sum_{i \neq j}^{M} {b_{ni}}^T \big(t\, Y + (1-t)\, Z \big)_i^T \big(t\, Y + (1-t)\, Z \big)_j b_{nj} \big)^2 \\
& - 2\epsilon \big(\sum_{i \neq j}^{M} {b_{ni}}^T \big(t\, Y + (1-t)\, Z \big)_i^T \big(t\, Y + (1-t)\, Z \big)_j b_{nj} \big) \big] \\
& + 2\mathrm{tr}({C^0}^T C^0) + t\frac{\rho}{2}||Y||_F^2 + t\frac{\rho}{2}||Z||_F^2 - t\rho||Y||_F||C^0||_F - (1-t)\rho||Z||_F||C^0||_F \\
& - t^2\, \frac{\rho}{2}||Y||_F^2 - t^2\, \frac{\rho}{2}||Z||_F^2 - \rho\sqrt{t}\sqrt{1-t}||Y||_F||Z||_F + \rho\sqrt{t}||Y||_F||C^0||_F + \rho\sqrt{1-t}||Z||_F||C^0||_F \\
= & \, t||Y||_F^2 + t||Z||_F^2 - 2t\mathrm{tr}(Y^T Z) \\
& - t^2||Y||_F^2 - t^2||Z||_F^2 + 2t^2\mathrm{tr}(Y^T Z) \\
& + 2\mathrm{tr}({C^0}^T C^0) \\
& + t\frac{\rho}{2}||Y||_F^2 - t^2\, \frac{\rho}{2}||Y||_F^2 + t\mu \sum_{n=1}^{N} \big[\big(\sum_{i \neq j}^{M} {b_{ni}}^T Y_i^T Y_j^k b_{nj} \big)^2 - 2\epsilon \big(\sum_{i \neq j}^{M} {b_{ni}}^T Y_i^T Y_j^k b_{nj} \big) \big] \\
& + t\frac{\rho}{2}||Z||_F^2 - t^2\, \frac{\rho}{2}||Z||_F^2 + (1-t)\mu \sum_{n=1}^{N} \big[\big(\sum_{i \neq j}^{M} {b_{ni}}^T Z_i^T Z_j^k b_{nj} \big)^2 - 2\epsilon \big(\sum_{i \neq j}^{M} {b_{ni}}^T Z_i^T Z_j^k b_{nj} \big) \big] \\
& - \mu \sum_{n=1}^{N} \big[\big(\sum_{i \neq j}^{M} {b_{ni}}^T \big(t\, Y + (1-t)\, Z \big)_i^T \big(t\, Y + (1-t)\, Z \big)_j b_{nj} \big)^2 \\
& - 2\epsilon \big(\sum_{i \neq j}^{M} {b_{ni}}^T \big(t\, Y + (1-t)\, Z \big)_i^T \big(t\, Y + (1-t)\, Z \big)_j b_{nj} \big) \big] \\
& - \rho\sqrt{t}\sqrt{1-t}||Y||_F||Z||_F \\
& + \sqrt{t}\rho||Y||_F||C^0||_F - t\rho||Y||_F||C^0||_F
\end{aligned} \tag{35}
$$

Since $t \in [0,1]$, we have $\sqrt{t} \geq t$ and $t \geq t^2$. Therefore,

$$\begin{aligned}
& t\phi(Y) + (1-t)\phi(Z) - \phi(tY + (1-t)Z) \\
\geq\ & t||Y||_F^2 + t||Z||_F^2 - 2t\mathrm{tr}(Y^T Z) \\
& - t^2||Y||_F^2 - t^2||Z||_F^2 + 2t^2\mathrm{tr}(Y^T Z) \\
& + 2\mathrm{tr}({C^0}^T C^0) \\
& t\mu \sum_{n=1}^{N} \big[\big(\sum_{i \neq j}^{M} {b_{ni}}^T {Y_i}^T Y_j^k b_{nj} \big)^2 - 2\epsilon \big(\sum_{i \neq j}^{M} {b_{ni}}^T {Y_i}^T Y_j^k b_{nj} \big) \big] \\
& (1-t)\mu \sum_{n=1}^{N} \big[\big(\sum_{i \neq j}^{M} {b_{ni}}^T {Z_i}^T Z_j^k b_{nj} \big)^2 - 2\epsilon \big(\sum_{i \neq j}^{M} {b_{ni}}^T {Z_i}^T Z_j^k b_{nj} \big) \big] \\
& - \mu \sum_{n=1}^{N} \big[\big(\sum_{i \neq j}^{M} {b_{ni}}^T \big(t\,Y + (1-t)\,Z\big)_i^T \big(t\,Y + (1-t)\,Z\big)_j b_{nj} \big)^2 \\
& - 2\epsilon \big(\sum_{i \neq j}^{M} {b_{ni}}^T \big(t\,Y + (1-t)\,Z\big)_i^T \big(t\,Y + (1-t)\,Z\big)_j b_{nj} \big) \big]
\end{aligned} \tag{36}$$

With Equation 18, 19 and 20, Equation 36 is further relaxed as:

$$\begin{aligned}
& t\phi(Y) + (1-t)\phi(Z) - \phi(tY + (1-t)Z) \\
\geq\ & t||Y||_F^2 + t||Z||_F^2 - 2t\mathrm{tr}(Y^T Z) \\
& - t^2||Y||_F^2 - t^2||Z||_F^2 + 2t^2\mathrm{tr}(Y^T Z) \\
& + 2dMK\mathcal{U}^2 - 4\mu dM^2\mathcal{U}^2
\end{aligned} \tag{37}$$

Now let us turn to the denominator of the left hand of Equation 32, we have

$$t(1-t)||Y-Z||_F^2 = t(1-t)\Big[||Y||_F^2 + ||Z||_F^2 - 2\mathrm{tr}(Y^T Z)\Big] \tag{38}$$

Put Equation 37 and 38 together, we have:

$$\begin{aligned}
& 2\frac{t\phi(Y) + (1-t)\phi(Z) - \phi(tY + (1-t)Z)}{t(1-t)||Y-Z||_F^2} \\
\geq\ & \frac{t\Big(||Y||_F^2 + ||Z||_F^2 - 2\mathrm{tr}(Y^T Z)\Big) - t^2\Big(||Y||_F^2 + ||Z||_F^2 - 2\mathrm{tr}(Y^T Z)\Big) + 2dMK\mathcal{U}^2 - 4\mu dM^2\mathcal{U}^2}{t(1-t)\Big[||Y||_F^2 + ||Z||_F^2 - 2\mathrm{tr}(Y^T Z)\Big]} \\
\geq\ & \frac{t(1-t)\Big(||Y||_F^2 + ||Z||_F^2 - 2\mathrm{tr}(Y^T Z)\Big) + 2dMK\mathcal{U}^2 - 4\mu dM^2\mathcal{U}^2}{t(1-t)\Big[||Y||_F^2 + ||Z||_F^2 - 2\mathrm{tr}(Y^T Z)\Big]} \\
\geq\ & 1 + \frac{2dMK\mathcal{U}^2 - 4\mu dM^2\mathcal{U}^2}{t(1-t)\Big[||Y||_F^2 + ||Z||_F^2 - 2\mathrm{tr}(Y^T Z)\Big]}
\end{aligned} \tag{39}$$

For the second term in last row of Equation 39, we would like note that the μ in our AAAI submission is only 0.0001, hence $K > 2\mu M$, and $2dMK\mathcal{U}^2 > 4\mu dM^2\mathcal{U}^2$. Therefore, the entire term will be positive, and we may simply set $m = 1$ to satisfy Inequality 32 for Assumption 2(a).

In the original Composite Quantization, it is also observed that small μ will not hurt the search performance. See Section 3.2 of [19] for details.

As a quasi-Newton method, L-BFGS [11, 10] can always find the optimal solution for a strongly convex problem, thus, each $\phi(C)$ will be exactly solved as is stated in Section 1.

The modulus

In mathematics, the modulus of convexity and the characteristic of convexity are measures of "how convex" the unit ball in a Banach space is. The modulus γ belongs to $[0, 1]$ by definition [1].

4.3 Proof of Assumption 2(b)

The assumption 2(b) is:

- For all k, the penalty parameter ρ_k is chosen large enough such that:
  - For all k, $\rho \geq L_k$ and $\rho\gamma_k(\rho) > 2{L_k}^2$.

In line #606 of our AAAI submission, we state that we set $\rho = 100$, which do not satisfy $\rho \geq L_k$ in most cases. Take the training set of SIFT1M dataset as an example, $N = 10^5, mu = 0.0001, d = 128, M = 8, \mathcal{U} = 197$, and the required $L = 7.19 \times 10^{10}$, which is much larger than our ρ.

The second condition $\rho\gamma_k(\rho) > 2{L_k}^2$ is clearly not satisfied either, even we suppose the modulus $\gamma(\rho)$ to be 1, which is the maximum of modulus.

However, we would like to emphasize that the relaxation in Section 4.1 is very large, and Equation 27 is a very loose upper bound. It is just a sufficient condition, rather than necessary condition.

In addition, we tried to set $\rho = L$ to train the quantizers in our AAAI algorithm. On SIFT1M dataset with 64 bits, we observe that the algorithm takes more iteration to converge and result in worse accuracy as shown in the table below:

ρ	10	**100**	10^4	10^6	10^8	10^{10}	10^{12}
#Iteration to Converge	20	**20**	23	31	38	50	67
Recall@1	0.273	**0.280**	0.264	0.218	0.185	0.142	0.107

We can observe that $\rho = 100$ leads to the best accuracy. Although $\rho = 100$ is less than the threshold $L = 7.19 \times 10^{10}$, we cannot conclude that our algorithm will not converge with $\rho = 100$ due to the fact that $L = 7.19 \times 10^{10}$ is a extremely loose bound. On the other hand, the $\rho = 10^{12}$ results in more training iteration and worse recall, even though it is provably convergent.

[1] `https://en.wikipedia.org/wiki/Modulus_and_characteristic_of_convexity`

4.4 Proof of Assumption 3

The objective function in Equation 5 consists of two terms. The first term is sum of Frobenius norm, the second term is sum of square. Therefore, the objective function in Equation 5 is larger or equal to 0, which satisfies the third assumption.

5 Conclusion

In the proof above, we reformulate the problem in our AAAI submission into the form of nonconvex global consensus problem in Section 3. In Section 4.1, we prove that there does exist a positive constant $L > 0$ for any matrix Y and Z such that

$$||\nabla g(Y) - \nabla g(Z)||_F \leq L||Y - Z||_F \tag{40}$$

In Section 4.2, we prove that all sub-problems of our augmented Lagrangian are strongly convex. In Section 4.4, we show that our objective function Equation 8 is lower-bounded.

Although our $\rho = 100$ does not satisfy Assumption 2(b), we would like to emphasize that Assumption 2(b) is sufficient but not necessary, due to the fact that the L in Section 4.1 in an extremely loose bound. As discussed in Section 4.2, $\rho = 100$ has better convergence speed and accuracy than $\rho = 10^{12}$, although $\rho = 10^{12}$ is provably convergent.

Finally, we would like to note that many researchers have observed that the ADMM works extremely well for various applications involving nonconvex objectives, such as the nonnegative matrix factorization [20, 14], phase retrieval [16], distributed matrix factorization [18], distributed clustering [3], sparse zero variance discriminant analysis [1], polynomial optimization [6], tensor decomposition [8], matrix separation [13], matrix completion [17], asset allocation [15], sparse feedback control [9] and so on.

References

[1] B. P. Ames and M. Hong. Alternating direction method of multipliers for penalized zero-variance discriminant analysis. *Computational Optimization and Applications*, 64(3):725–754, 2016.

[2] S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein. Distributed optimization and statistical learning via the alternating direction method of multipliers. *Foundations and Trends® in Machine Learning*, 3(1):1–122, 2011.

[3] P. A. Forero, A. Cano, and G. B. Giannakis. Distributed clustering using wireless sensor networks. *IEEE Journal of Selected Topics in Signal Processing*, 5(4):707–724, 2011.

[4] M. Hong, Z.-Q. Luo, and M. Razaviyayn. Convergence analysis of alternating direction method of multipliers for a family of nonconvex problems. *SIAM Journal on Optimization*, 26(1):337–364, 2016.

[5] H. Jégou, R. Tavenard, M. Douze, and L. Amsaleg. Searching in one billion vectors: re-rank with source coding. In *Acoustics, Speech and Signal Processing (ICASSP), 2011 IEEE International Conference on*, pages 861–864. IEEE, 2011.

[6] B. Jiang, S. Ma, and S. Zhang. Alternating direction method of multipliers for real and complex polynomial optimization models. *Optimization*, 63(6):883–898, 2014.

[7] Y. LeCun, L. Bottou, Y. Bengio, and P. Haffner. Gradient-based learning applied to document recognition. *Proceedings of the IEEE*, 86(11):2278–2324, 1998.

[8] A. P. Liavas and N. D. Sidiropoulos. Parallel algorithms for constrained tensor factorization via alternating direction method of multipliers. *IEEE Transactions on Signal Processing*, 63(20):5450–5463, 2015.

[9] F. Lin, M. Fardad, and M. R. Jovanović. Design of optimal sparse feedback gains via the alternating direction method of multipliers. *IEEE Transactions on Automatic Control*, 58(9):2426–2431, 2013.

[10] D. C. Liu and J. Nocedal. On the limited memory bfgs method for large scale optimization. *Mathematical programming*, 45(1):503–528, 1989.

[11] J. Nocedal. Updating quasi-newton matrices with limited storage. *Mathematics of computation*, 35(151):773–782, 1980.

[12] B. C. Russell, A. Torralba, K. P. Murphy, and W. T. Freeman. Labelme: a database and web-based tool for image annotation. *International journal of computer vision*, 77(1-3):157–173, 2008.

[13] Y. Shen, Z. Wen, and Y. Zhang. Augmented lagrangian alternating direction method for matrix separation based on low-rank factorization. *Optimization Methods and Software*, 29(2):239–263, 2014.

[14] D. L. Sun and C. Fevotte. Alternating direction method of multipliers for non-negative matrix factorization with the beta-divergence. In *Acoustics, Speech and Signal Processing (ICASSP), 2014 IEEE International Conference on*, pages 6201–6205. IEEE, 2014.

[15] Z. Wen, X. Peng, X. Liu, X. Sun, and X. Bai. Asset allocation under the basel accord risk measures. *arXiv preprint arXiv:1308.1321*, 2013.

[16] Z. Wen, C. Yang, X. Liu, and S. Marchesini. Alternating direction methods for classical and ptychographic phase retrieval. *Inverse Problems*, 28(11):115010, 2012.

[17] Y. Xu, W. Yin, Z. Wen, and Y. Zhang. An alternating direction algorithm for matrix completion with nonnegative factors. *Frontiers of Mathematics in China*, 7(2):365–384, 2012.

[18] R. Zhang and J. Kwok. Asynchronous distributed admm for consensus optimization. In *Proceedings of the 31st International Conference on Machine Learning (ICML-14)*, pages 1701–1709, 2014.

[19] T. Zhang, C. Du, and J. Wang. Composite quantization for approximate nearest neighbor search. In *Proceedings of the 31st International Conference on Machine Learning (ICML-14)*, pages 838–846, 2014.

[20] Y. Zhang. An alternating direction algorithm for nonnegative matrix factorization. 2010.